PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON TUESDAY, MARCH 19, 2024, ON SOLUTION CURVES of a D.E., The D.E.: $P' = kP$, and an Application of a SEPARABLE DIFFERENTIAL EQUATION

CLASS #18.

Differential Equations

EXAMPLE: $y'' + 4y = 0$ for $y = g(x)$

The general solution for this D.E. is

$$y = C_1 \cos(2x) + C_2 \sin(2x),$$

C_1 and C_2 are constants.

One solution of this D.E. comes from $C_1 = 0$ and $C_2 = 7$ giving

$$y = 7 \sin(2x).$$

You can check that this is a solution.

An <u>Analytic</u> <u>Solution</u> $\underline{y = f(x)}$

Initial Value Problems (IVPs)

Example of An IVP:

$y'' + 4y = 0$ is a given D.E. with $y = y(x)$.

The Initial Conditions → FIND a solution ~~a~~ solution of D.E. such that
$y(\pi/4) = 3$ and $y'(\pi/4) = 10$.

How we solved it:

The General Solution of the D.E is

$$y = C_1 \cos(2x) + C_2 \sin(2x).$$

$$y' = -2C_1 \sin(2x) + 2C_2 \cos(2x)$$

We evaluated both at $x = \pi/4$ to get equation ~~eq~~ in C_1 and C_2, and we solved these to find $C_1 = -5$ and $C_2 = 3$.

So, the Solution of the IVP is

$$y = -5\cos(2x) + 3\sin(2x).$$

Consider the D.E., $4yy' - 2x = 0$

The General Solution is

$$2y^2 - x^2 = C \text{ where } -\infty < C < \infty.$$

(Unfortunately, it only defines solutions y implicitly. This is an example of a non-analytic solution.)

Check: Differentiate both sides to get:

$$2(2yy') - 2x = 0$$

$$4yy' - 2x = 0 \quad \text{which is the D.E.}$$

The graph of a particular solution for a D.E. is called a solution curve of the D.E.,

The Graphs of all the solutions of a D.E. are called "The Solution Curves of the D.E."

Now, Consider the D.E. $4yy' - 2x = 0$.

Task: Find the solution curves of this D.E.

The General Solution of this D.E. is

$$2y^2 - x^2 = C, \quad -\infty < C < \infty$$

For $C = 0$,

$$2y^2 - x^2 = 0$$

$$2y^2 = x^2$$

$$\sqrt{2}\,|y| = |x|$$

$$|y| = \frac{1}{\sqrt{2}}|x|$$

$$y = \pm\frac{1}{\sqrt{2}}|x|$$

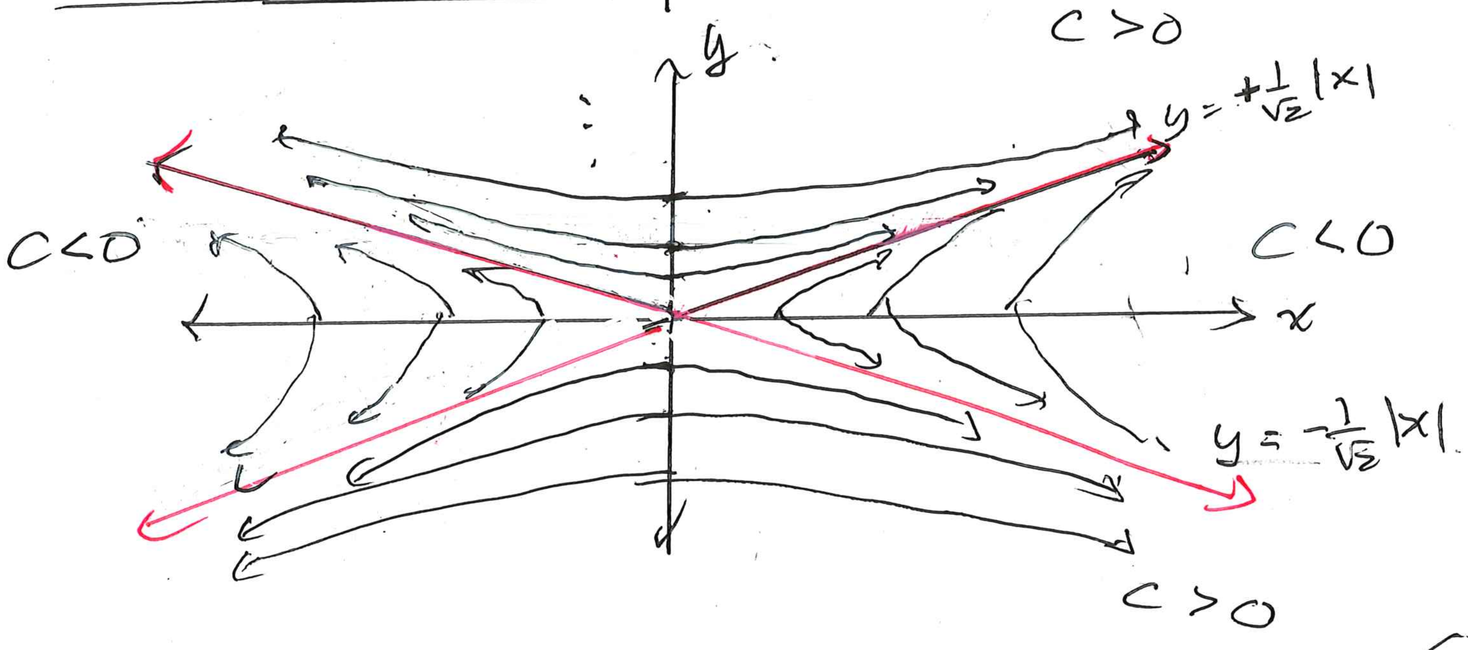
$$y = \pm 0.7071|x|$$

For $C \neq 0$,

$$\frac{2y^2}{C} - \frac{x^2}{C} = 1$$

$$\frac{y^2}{\left(\frac{C}{2}\right)} - \frac{x^2}{C} = 1$$

Hyperbolas!



EXAMPLES OF FINDING A General Solution of a Differential

Ex: Let k be a non-zero constant real number.

Consider the D.E. $\dfrac{dP}{dt} = kP$

(Here $P = f(t) = P(t)$)

FIND the General Solution of this D.E.

<u>Solution</u>: Omitting, for now, the zero function $P(t) = 0$

and assuming $P \neq$ the zero function,

divide both sides by P to get:

$$\frac{P'}{P} = \frac{dP/dt}{P} = k \Rightarrow \int \frac{P'(t)}{P(t)}\,dt = \int k\,dt.$$

$$\int \frac{P'(t)}{P(t)}\,dt = \int k\,dt = kt + C,$$ where C is any real #.

$$\int \frac{P'(t)}{P(t)}\,dt = \ln(|P(t)|) = kt + C,$$ where C is any real #.

(Apply e^x to both sides)

$$|P(t)| = e^{(kt+C)} = e^{kt} \cdot e^{C}$$ and Note $e^C > 0$

Let $A = e^C > 0$

$$|P(t)| = Ae^{kt}$$ where A is any positive real #. $A > 0$.

$|P(t)| = Ae^{kt}$ where $A > 0$.

$P(t) = \pm A e^{kt}$ where A is any positive real #.
$\pm A \neq 0$.

Since $\pm A$ can be any non-zero number,
I can redefine A as $A \neq 0$, so:

$P(t) = Ae^{kt}$ where A is any non-zero #;
$A \neq 0$

(This is a general solution for all the Non-Zero Sol'ns of the D.E.).

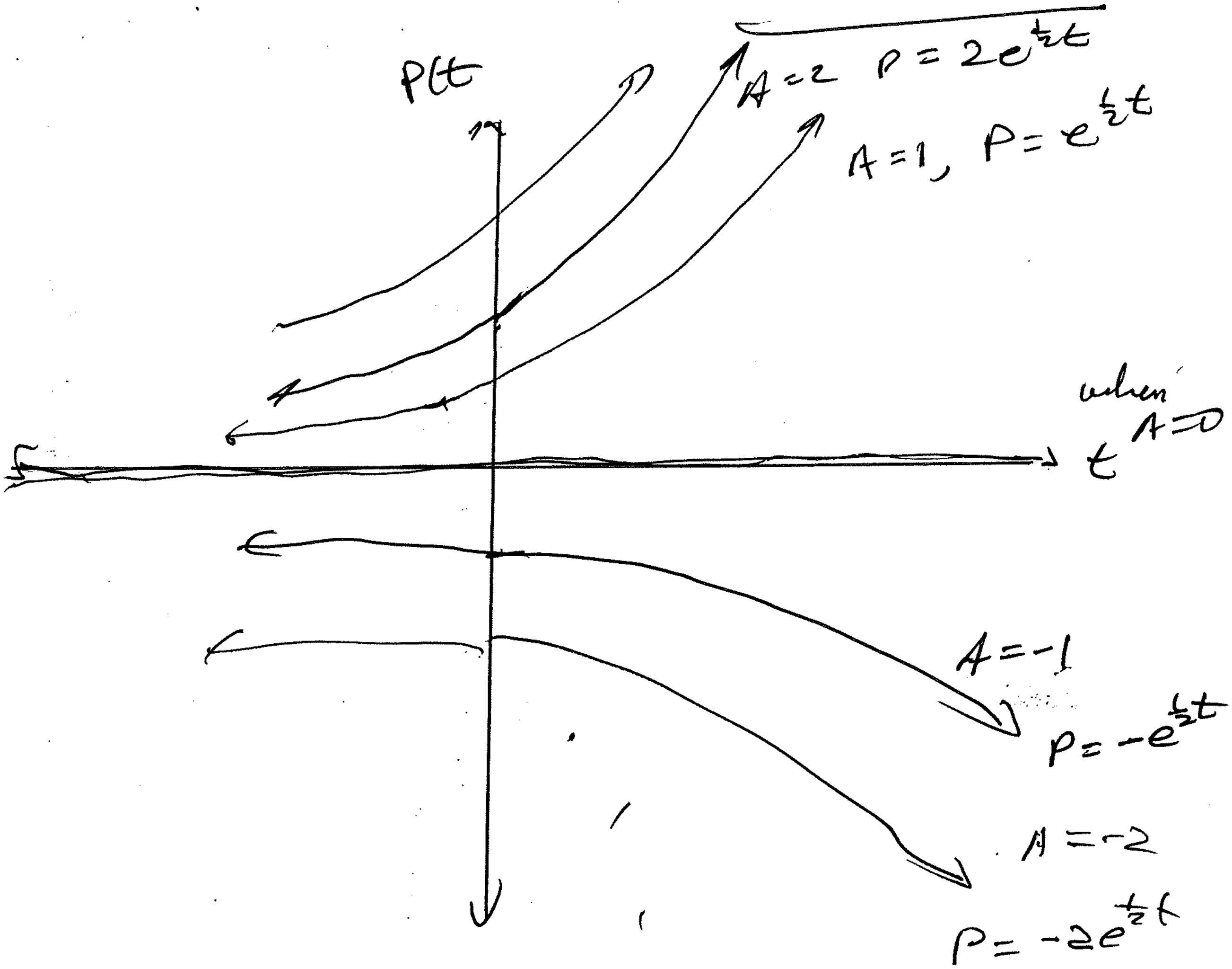
Is $P(t) = 0$ a solution too?,
For $P(t) = 0$, $P'(t) = 0$, and $P'(t) = k \cdot 0 = 0$
$= 0$
and $kP(t) = 0$ ∴ so, $P'(t) = kP(t)$.

So $P(t) = 0$ is a solution of the D.
$P(t) = 0 = 0 \cdot e^{kt}$ so Ae^{kt} when $A = 0$ is a sol'n.

The General Solution of the D.E. $P'(t) = kP(t)$
is $P(t) = Ae^{kt}$ where A is any real number
(including $A = 0$)

For $k = \frac{1}{2}$, The D.E. is $P' = \frac{1}{2}P$

and it has General solution $P = Ae^{\frac{1}{2}t}$.



THESE ARE THE SOLUTION CURVES FOR THE D.E.

$$P' = \frac{1}{2}P$$

A Separable First-Order Differential Equation Application

A full 200-gallon tank has brine (salt water) with 25 lbs of salt in the tank. Starting at time $t = 0$ min, a brine solution with concentration 0.05 lbs / gal enters the tank at the constant rate of 10 gal/min, and the well-stirred solution is drained from the tank at the same rate, 10 gal/min.

(a) Determine the amount of salt in the tank as a function of time **t**.

(b) When does the amount of salt in the tank become 15 lbs ?

Solution: Let $y(t)$ = # of lbs of salt in the tank at time t minutes. The Initial Condition is $y(0) = 25$.

We Discover a Differential EQUATION in function y which has the function $y(t)$ as a solution.

$$\frac{dy}{dt} = \text{RATE}_{\text{IN}} - \text{RATE}_{\text{OUT}} \quad \text{in terms of } \frac{\text{lbs of SALT}}{\text{MIN}}.$$

$$\text{RATE}_{\text{IN}} = \underbrace{(0.05\ \text{lbs/gal})}_{\text{BRINE CONCENTRATION}} \times \underbrace{(10\ \text{gal/min})}_{\text{RATE OF FLOW}} = 0.5\ \text{lbs of salt/min}.$$

$$\text{RATE}_{\text{OUT}} = \underbrace{\left(\frac{y(t)\ \text{lbs}}{200\ \text{gal}}\right)}_{\text{BRINE CONCENTRATION}} \times \underbrace{(10\ \text{gal/min})}_{\text{RATE OF FLOW}} = \frac{y}{20}\ \text{lbs of salt/min}$$

$$\frac{dy}{dt} = 0.5 - \frac{y}{20} = \frac{10}{20} - \frac{y}{20} = \frac{1}{20}(10 - y)$$

The Initial Condition is $y(0) = 25$.

So, the I.V.P. to solve is:

$$\frac{dy}{dt} = \frac{1}{20}(10-y) \text{ and } y(0) = 25.$$

$$\frac{1}{10-y}\,dy = \frac{1}{20}\,dt$$

$$\int \frac{1}{10-y}\,dy = \int \frac{1}{20}\,dt = \frac{1}{20}t + C_1$$

$$= -\int \frac{-1}{10-y}\,dy = -\ln|10-y| + C_2$$

$$-\ln|10-y| = \frac{1}{20}t + C_3 \qquad (C_3 = C_1 - C_2)$$

$$\ln|10-y| = C - \frac{1}{20}t \qquad C = -C_3$$

[TAKING EXPONENTIALS]

$$|10-y| = e^{C} \cdot e^{-\frac{1}{20}t}$$

[we solve for e^{C}] So at $t=0$, $y = 25$ lbs of salt, so

$$|10-25| = e^{C} \cdot e^{-0/20} = e^{C} \cdot e^{0} = e^{C}$$

$(|-15| = 15)$

$$15 = e^{C}$$

$(|10-y| = |y-10|)$

$$\text{So, } |y-10| = 15e^{-\frac{1}{20}t}$$

$$y - 10 = \pm 15e^{-t/20}$$

$$y = 10 \pm 15e^{-t/20}. \qquad \text{Since } y(0) = 25 > 0, \text{ use } +, \text{ not } -.$$

Part (a) SOLUTION: $y = 10 + 15e^{-t/20}$ lbs of salt in the tank at time t_{min}

Part (b) sol'n: Set $y = 15$ and solve for t.

$$15 = 10 + 15e^{-t/20} \Rightarrow 5 = 15e^{-t/20} \Rightarrow e^{-t/20} = \frac{1}{3} \Rightarrow -\frac{t}{20} = \ln\left(\frac{1}{3}\right)$$

$$\Rightarrow -t/20 = -\ln 3 \Rightarrow \boxed{t = 20\ln 3 \approx 21.97 \text{ minutes}}$$